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Quadratic gravity in (2+1)D with a topological Chern–Simons term

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Abstract

Three-dimensional quadratic gravity, unlike general relativity in (2+1)D, is dynamically nontrivial and has a well behaved nonrelativistic potential. Here we analyse the changes that occur when a topological Chern–Simons term is added to this theory. It is found that the harmless massive scalar mode of the latter gives rise to a troublesome massive spin-0 ghost, while the massive spin-2 ghost is replaced by two massive physical particles both of spin 2. We also found that light deflection does not have the 'wrong sign' such as in the framework of three-dimensional quadratic gravity.

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1. Introduction

As is well known, Einstein's equations also hold in three-dimensional space–time. However, the nature of gravity is quite different from that in four-dimensional space–time [1]. Indeed, three-dimensional general relativity is a generally covariant theory that has no propagating gravitational degrees of freedom, because in the absence of sources it is solved uniquely by flat space–time [2]. Curvature is created by sources, but only locally at their positions; elsewhere space–time remains flat. Consequently, there are no gravitons, and forces are not mediated by graviton exchange; rather, they are geometrical/topological in origin, arising from global properties of space–time, which is not Minkowskian in the large, even when it is locally flat [3,4].

When the four-derivative terms $\int R^2_{\mu\nu}\sqrt{g}\,\mathrm{d}^3x$ and $\int R^2\sqrt{g}\,\mathrm{d}^3x$ are included into three-dimensional Einstein action, some significant changes occur. In this case, we have a class of effectively multimass models of gravity. In addition to the massless excitations of the field which, incidentally, are non-propagating [5] such as in three-dimensional general relativity, there are now—for the general amount of the two new terms—massive spin-2 and massive scalar excitations. Quadratic gravity in (2+1)D is also locally nontrivial and has an extremely well behaved potential [6]. Besides, within the context of the same a gravitational force is exerted on a slowly moving test particle [7]. The expression for this force greatly resembles that of a straight $U(1)$-gauge cosmic string in the framework of linearized quadratic gravity in four-dimensions [8]. On the other hand, light deflection has the 'wrong sign' in the framework of this theory [9]. In other words, a light ray is deflected upward instead of downward as its (3+1)D counterpart.

Our aim here is to examine how these properties are modified when a topological Chern–Simons term is added to the Lagrangian related to (2+1)-dimensional quadratic gravity.

The plan of this work is as follows. In section 2 we present a discussion on the power-counting renormalizability of the above theory. We also find the expression concerning its nonrelativistic potential. In section 3 we show that the massless excitation is not a dynamical degree of freedom. We study, in section 4, the gravitational force exerted on a slowly moving test particle as well as the deflections of light rays. Final remarks comprise section 5. In our notation the signature is $(+--)$. the curvature tensor is defined by $R^{\alpha}{}_{\beta\gamma\delta} = -\partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \cdots$ and the Ricci tensor by $R = g^{\mu\nu}R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor. Natural units are used throughout.

2. Causality, power-counting renormalizability and effective nonrelativistic potential

The Lagrangian for quadratic-Chern–Simons gravity in (2+1)D can be cast in the form

$$\mathcal{L} = -\frac{2R\sqrt{g}}{\kappa^2} + \frac{\varepsilon^{\mu\nu\lambda}}{\mu}\Gamma^{\rho}{}_{\sigma\lambda}\left(\partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\omega\mu}\Gamma^{\omega}{}_{\nu\sigma}\right) + \left(\frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right)\sqrt{g} \tag{1}$$

where α and β are constants with dimension L, κ^2 is a suitable constant with dimension L which is not necessarily related to the Einstein's constant in four dimensions and μ is a dimensionless parameter. Here $\varepsilon^{012} = +1$. To find the graviton propagator we need the bilinear part of (1). This is obtained by decomposing the metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and inserting (2) into (1). Let $\bar{\mathcal{L}}$ be the resulting Lagrangian. Since our theory is gauge-invariant, we add to $\bar{\mathcal{L}}$ the gauge-fixing Lagrangian

$$\mathcal{L}_{\text{gf}} = -\lambda_1 A^2_\nu - \frac{b}{4}\left[\lambda_2 \left(A^{\mu}{}_{,\mu}\right)^2 + \lambda_3 F^2_{\mu\nu}\right]$$

where λ_1, λ_2 and λ_3 are gauge-parameters, $A^\mu \equiv h^{\mu\nu}_{\ ,\nu}$, $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$ and $b \equiv \frac{\beta\kappa^2}{2}$. This is the Lagrangian corresponding to the Julve–Tonin gauge [10]. Therefore, $\bar{\bar{\mathcal{L}}} \equiv \bar{\mathcal{L}} + \mathcal{L}_{\text{gf}}$ can be written as $\bar{\bar{\mathcal{L}}} = h^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma}$. Expanding now the operator $\mathcal{O}$ in the basis $\{P^1, P^2, P^0, \bar{P}^0, \bar{\bar{P}}^0, P\}$, whereupon [11]

$$P_{\mu\nu,\rho\sigma} \equiv \frac{\Box\,\partial^\lambda}{4}\left[\varepsilon_{\mu\lambda\rho}\theta_{\nu\sigma} + \varepsilon_{\mu\lambda\sigma}\theta_{\nu\rho} + \varepsilon_{\nu\lambda\rho}\theta_{\mu\sigma} + \varepsilon_{\nu\lambda\sigma}\theta_{\mu\rho}\right]$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is the usual transverse vector projection operator and P^1, P^2, P^0, $\bar{P}^0$, $\bar{\bar{P}}^0$ are the three-dimensional Barnes–Rivers operators [12], we obtain

$$\mathcal{O} = -k^2\left(\lambda_1 + \lambda_3\frac{b}{2}k^2\right)P^1 + k^2\left(k^2\frac{b}{2} - 1\right)P^2 + \left[k^2 + bk^4\left(\frac{3}{2} + 4c\right)\right]P^0 - k^2\left(\frac{b}{2}\lambda_2 k^2 + 2\lambda_1\right)\bar{P}^0 + \frac{P}{M}.$$

Here $c \equiv \frac{\alpha}{\beta}$ and $M \equiv \frac{\mu}{\kappa^2}$. The propagator is then given by

$$\mathcal{O}^{-1} = \frac{-2}{k^2\left[2\lambda_1 + b\lambda_3 k^2\right]} P^1 + \left[-\frac{1}{k^2} + \frac{1}{1+\frac{1}{2}bM_2^2}\,\frac{1}{k^2 - M_2^2} + \frac{1}{1+\frac{1}{2}bM_1^2}\,\frac{1}{k^2 - M_1^2}\right] P^2$$
$$+ \left[\frac{1}{k^2} - \frac{1}{k^2 - m^2}\right] P^0 - \frac{1}{k^2\left[2\lambda_1 + \lambda_2 \frac{b}{2} k^2\right]} \bar{P}^0$$
$$- \left[\frac{4}{b^2 M\left(M_1^2 - M_2^2\right)} \left(\frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2}\right) \frac{1}{k^4}\right] P \tag{3}$$

where

$$M_1^2 \equiv \left(\frac{2}{b^2 M^2}\right)\left[1 + bM^2 + \sqrt{1 + 2bM^2}\right]$$
$$M_2^2 \equiv \left(\frac{2}{b^2 M^2}\right)\left[1 + bM^2 - \sqrt{1 + 2bM^2}\right]$$
$$m^2 \equiv \frac{-1}{b\left(\frac{3}{2} + 4c\right)}.$$

If we do not want tachyons in the dynamical field we may choose, for instance, $b > 0$ and $\left(\frac{3}{2} + 4c\right) < 0$. In this case the theory is causal at the tree level. In this vein we assume from now on m^2, M_1^2, M_2^2 and $M^2 > 0$.

Since the coefficients of P^2 and P^0 in (3) behave asymptotically as k^{-6} and k^{-4} respectively, quadratic-Chern–Simons gravity is power-counting renormalizable.

The effective nonrelativistic potential for this model can be computed from the expression

$$U(\boldsymbol{r}) = \frac{1}{4\tilde{m}^2}\,\frac{1}{(2\pi)^2}\int \mathrm{d}^2\boldsymbol{k}\; \mathcal{M}_{\mathrm{NR}}\; \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{4}$$

where $\mathcal{M}_{\mathrm{NR}}$ is the nonrelativistic limit of the Feynman amplitude for the process $S+S \to S+S$, where S denotes a spinless boson of mass $\tilde{m}$. The invariant amplitude for this process is given by

$$\mathcal{M} = V^{\mu\nu}\left(p, -p'\right) \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} V^{\rho\sigma}\left(q, -q'\right)$$

where

$$V_{\mu\nu}\left(p, p'\right) = \frac{1}{2}\kappa\left[p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}\left(p\cdot p' + \tilde{m}^2\right)\right]$$

is the vertex function for the trilinear coupling $\mathsf{g}(k) - \phi(p) - \phi(p')$. Here all momenta are supposed to be incoming. Thence,

$$\mathcal{M} = \kappa^2\left[-\frac{1}{2k^2} + \frac{1}{bM_2^2 + 2}\,\frac{1}{k^2 - M_2^2} + \frac{1}{b^2M_1^2 + 2}\,\frac{1}{k^2 - M_1^2}\right]$$
$$\times\Big[(p\cdot q)(p'\cdot q') + (p\cdot q')(p'\cdot q) + p\cdot p'\left(\tilde{m}^2 - q\cdot q'\right) + \left(\tilde{m}^2 - p\cdot p'\right) q\cdot q'$$
$$+ \frac{3}{2}\left(\tilde{m}^2 - q\cdot q'\right)\left(\tilde{m}^2 - p\cdot p'\right) - \frac{1}{4}\left(3\tilde{m}^2 - p\cdot p'\right)\left(3\tilde{m}^2 - q\cdot q'\right)\Big]$$
$$+ \frac{\kappa^2}{8}\left\{\left(3\tilde{m}^2 - p\cdot p'\right)\left(3\tilde{m}^2 - q\cdot q'\right)\left[\frac{1}{k^2} - \frac{1}{k^2 - m^2}\right]\right\}$$

and

$$\mathcal{M}_{\mathrm{NR}} = \kappa^2\tilde{m}^4\left[\frac{1}{2}\,\frac{1}{\boldsymbol{k}^2 + m^2} - \frac{1}{2 + bM_2^2}\,\frac{1}{\boldsymbol{k}^2 + M_2^2} - \frac{1}{2 + bM_1^2}\,\frac{1}{\boldsymbol{k}^2 + M_1^2}\right]. \tag{5}$$

Inserting (5) into (4) and performing the integration yields

$$U(r) = 2\tilde{m}^2\bar{G}\left[K_0(rm) - \frac{1}{1+\frac{1}{2}bM_1^2}K_0(rM_1) - \frac{1}{1+\frac{1}{2}bM_2^2}K_0(rM_2)\right]$$

where $\bar{G} \equiv \frac{\kappa^2}{32\pi}$ and K_0 is the modified Bessel function of the order of zero.

Consequently, the potential is given by the expression

$$V(r) = 2\tilde{m}\bar{G}\left[K_0(rm) - \frac{1}{1+\frac{1}{2}bM_1^2}K_0(rM_1) - \frac{1}{1+\frac{1}{2}bM_2^2}K_0(rM_2)\right].$$

Note that $V(r)$ behaves as

$$2\tilde{m}\bar{G}\ln\left(\frac{M_1^{1+\frac{1}{2}bM_1^2}M_2^{1+\frac{1}{2}bM_2^2}}{m}\right)$$

at the origin and as

$$2\tilde{m}\bar{G}\left[\sqrt{\frac{\pi}{2mr}}\mathrm{e}^{-rm} - \frac{1}{1+\frac{1}{2}bM_1^2}\sqrt{\frac{\pi}{2M_1r}}\mathrm{e}^{-M_1r} - \frac{1}{1+\frac{1}{2}bM_2^2}\sqrt{\frac{\pi}{2M_2r}}\mathrm{e}^{-M_2r}\right]$$

asymptotically. Two comments are in order here:

(i) Unlike the Newtonian potential $V_N = 2G\tilde{m}\ln\frac{r_0}{r_1}$ which has a logarithmic singularity at the origin and is unbounded at infinity, the potential concerning quadratic-Chern–Simons gravity in (2+1)D is extremely well behaved: it is finite at the origin and zero at infinity.

(ii) $V(r) \to 0$ as α and $\beta \to 0$, confirming in this way the fact that the standard correspondence of Einstein–Chern–Simons gravity in 3D with Newton's theory breaks down [13,14].

And, of course,

$$\kappa h_{00} = 2V. \tag{6}$$

3. The massless excitation is non-propagating

Let us now show that the massless excitation is not a dynamical degree of freedom. To do that we couple the propagator to external conserved currents, $T^{\mu\nu}$, compatible with the symmetries of the theory and afterwards we compute the residue of the current-current amplitude at the pole $k^2 = 0$. The transition amplitude can be cast in the form

$$\mathcal{A} = \mathsf{g}^2 T^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}T^{\rho\sigma}$$

where g is the effective coupling constant of the theory. We expand now the sources in a suitable basis. The set of independent vectors in momentum space,

$$k^\mu \equiv (k^0, \boldsymbol{k}) \qquad \tilde{k}^\mu \equiv (k^0, -\boldsymbol{k}) \qquad \varepsilon^\mu \equiv (0, \vec{\epsilon})$$

where $\vec{\epsilon}$ is a unit vector orthogonal to $\boldsymbol{k}$, serves our purpose. Accordingly, the symmetric current $T^{\mu\nu}(k)$ can be written as

$$T^{\mu\nu} = ak^\mu k^\nu + b\tilde{k}^\mu\tilde{k}^\nu + c\varepsilon^\mu\varepsilon^\nu + dk^{(\mu}\tilde{k}^{\nu)} + ek^{(\mu}\varepsilon^{\nu)} + f\tilde{k}^{(\mu}\varepsilon^{\nu)}.$$

The current conservation, $k_\mu T^{\mu\nu} = 0$, gives the following constraints for the coeffeicients a, b, e and f:

$$ak^2 + \left(k_0^2 + \boldsymbol{k}^2\right)\frac{d}{2} = 0$$
$$b\left(k_0^2 + \boldsymbol{k}^2\right) + d\frac{k^2}{2} = 0$$
$$ek^2 + f\left(k_0^2 + \boldsymbol{k}^2\right) = 0.$$

If we saturate the indices of $T^{\mu\nu}$ with momenta k_μ, we obtain the equation $k_\mu k_\nu T^{\mu\nu} = 0$, which yields a consistency relation for the coefficients a, b and d

$$ak^4 + b\left(k_0^2 + \boldsymbol{k}^2\right)^2 + dk^2\left(k_0^2 + \boldsymbol{k}^2\right) = 0.$$

Hence,

$$\mathrm{Res}\mathcal{A}\mid_{k^2=0} = \mathsf{g}^2\left[c^2 - c^2\right]_{k^2=0} = 0$$

implying that the massless excitation is non-propagating.

4. Gravitational acceleration and gravitational deflection of light rays

Using the results found in the last two sections we shall now study the following issues:

(i) gravitational force exerted on a slowly moving test particle;
(ii) gravitational deflection of light rays.

4.1. Gravitational acceleration

In the weak-field approximation, the gravitational acceleration $\gamma^k = \mathrm{d}v^k/\mathrm{d}t$ of a slowly moving test particle is given by

$$\gamma^k = -\kappa\left[h^k_{0,0} - \frac{1}{2}h^{,k}_{00}\right].$$

Since our field is time independent, this equation reduces to

$$\gamma^k = -\frac{\kappa}{2}h_{00,k}. \tag{7}$$

Inserting (6) into (7), we get

$$\gamma^k = -2\bar{G}\tilde{m}\frac{x^k}{r}\left[\frac{M_1K_1(rM_1)}{1+\frac{bM_1^2}{2}} + \frac{M_2K_1(rM_2)}{1+\frac{bM_2^2}{2}} - mK_1(rm)\right]. \tag{8}$$

In the absence of the Chern–Simons term ($M \to \infty$), (8) reduces to

$$\gamma^k = -2\tilde{m}\tilde{G}\frac{x^k}{r}\left[NK_1(rN) - mK_1(rm)\right] \tag{9}$$

with $N^2 \equiv \frac{2}{b}$. Since $\mathrm{d}[xK_1(x)]/\mathrm{d}x = -xK_0(x)$ and $K_0(x) > 0$ it follows that $xK_1(x)$ is a positive monotonically function in the range $0 \leqslant x < \infty$. Accordingly, the gravitational force related to (9) is everywhere attractive if $N < m$, is repulsive if $N > m$ and vanishes if $N = m$. From the preceding analysis we come to the conclusion that 'antigravity' is possible in the framework of (2+1)D quadratic-Chern–Simons gravity.

4.2. Light deflection

Let us consider now the interaction between a fixed source and a light ray. The associated energy–momentum tensors will be designated respectively as $T^{\mu\nu}$ and $F^{\mu\nu}$. The current–current amplitude for this process is given by

$$\mathcal{A} = \mathrm{g}^2 T^{\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} F^{\mu\sigma}.$$

But on mass-shell $k_\mu T^{\mu\nu} = 0$ and $k_\mu F^{\mu\nu} = 0$, implying that only P^2 and P^0 will give a nonnull contribution to the current–current amplitude. Therefore

$$\mathcal{A} = \mathrm{g}^2 T^{\mu\nu} F^{\rho\sigma} \left[\left(-\frac{1}{k^2} + \frac{1}{1 + \frac{1}{2}bM_2^2} \frac{1}{k^2 - M_2^2} + \frac{1}{1 + \frac{1}{2}bM_1^2} \frac{1}{k^2 - M_1^2} \right) P^2 + \left(\frac{1}{k^2} - \frac{1}{k^2 - m^2} \right) P^0 \right]_{\mu\nu,\rho\sigma}.$$

Now, taking into account that [5]

$$P^2_{\mu\nu,\rho\sigma} = \frac{1}{2} \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma} \right) - \left[P^1 + \frac{1}{2}\bar{P}^0 - \frac{1}{2}\bar{\bar{P}}^0 \right]_{\mu\nu,\rho\sigma}$$

$$P^0_{\mu\nu,\rho\sigma} = \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{2} \left[\bar{P}^0 + \bar{\bar{P}}^0 \right]_{\mu\nu,\rho\sigma}$$

and recalling that the massless excitation is non-propagating, we promptly obtain

$$\mathcal{A} = \mathrm{g}^2 T_{00} F_{00} \left(\frac{1}{1 + \frac{1}{2}bM_2^2} \frac{1}{k^2 - M_2^2} + \frac{1}{1 + \frac{1}{2}bM_1^2} \frac{1}{k^2 - M_1^2} \right).$$

Hence, the light ray will be deflected downward as usual.

5. Final remarks

A comparison between three-dimensional quadratic gravity and quadratic-Chern–Simons gravity in (2+1)D shows that the harmless massive scalar mode of the former becomes a troublesome massive spin-0 ghost in the framework of the latter, while the massive spin-2 ghost related to quadratic gravity in (2+1)D is now replaced by two massive physical particles both of spin 2. On the other hand, light deflection has the ‘wrong sign’ within the context of three-dimensional higher-derivative gravity [9]. The addition of a topological massive term to the latter ‘repairs’ the aforementioned sign.

We list in the following some interesting features of the theory of quadratic gravity with a Chern–Simons term in (2+1)D:

(i) the nonrelativistic potential is extremely well behaved;
(ii) the massless excitation is non-propagating;
(iii) ‘antigravity’ is possible.

To conclude we raise an interesting question: Is the photon propagation dispersive in the framework of (2+1)D quadratic gravity with a Chern–Simons term such as in quadratic gravity in (3+1)D [15–17]? This issue will be discussed elsewhere.

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